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Integration on Loop Groups. I. Quasi Invariant Measures

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1. Introduction

On a riemannian manifold M, the Brownian motion defines a Wiener measure on the loops over M. In the case of $\mathbb{R}^n$, quasi invariance means the Cameron–Martin theorem [5] which fully describes the quasi invariance by translation of the Wiener measure.

In order to state a generalization of this theorem we suppose that M is an homogeneous space under the action of a *compact* Lie group. The non-commutativity of this situation introduces a completely new phenomenon: the hypoellipticity associated to the Lie bracket of the basic vector fields. Curiously enough hypoellipticity, which helps to get regularity in finite dimensions, works against quasi invariance in infinite dimensions [18].

We denote by G a *compact*, connected Lie group, by H a connected subgroup of G, and by M the homogeneous space G/H. We denote by $\mathscr{G}$ and $\mathscr{H}$ the corresponding Lie algebras. We choose a euclidean metric on $\mathscr{G}$ which is $\mathrm{Ad}(g)$ invariant. This metric induces a riemannian metric on G and on M.

We denote $\mathbb{P}_e(G)$ the paths space on G, that is, the space of continuous maps γ from $[0, 1]$ into G such that $\gamma(0)=e$.

We denote by $\mathbb{L}_e(G)$ the space of *loops* on G; that is,

$$\mathbb{L}_e(G)=\{\gamma\in\mathbb{P}_e(G);\ \gamma(1)=e\}.$$

Given $m_0\in M$, we define $\mathbb{P}_{m_0}(M)$ and $\mathbb{L}_{m_0}(M)$ in the same way.

If N is a riemannian manifold, we define *the Wiener measures* $\mu_{\mathbb{P}_{n_0}(N)}$, $\mu_{\mathbb{L}_{n_0}(N)}$ in the following way:

By $p_t(n, n')$ we denote the fundamental solution of the heat equation $(\partial/\partial t-\frac{1}{2}\Delta_N)$. Given a subdivision S, $0<t_0<t_1<t_{|S|}<1$ of $[0, 1]$, we denote by $\mathscr{E}_S$ the evaluation map from $\mathbb{P}_{n_0}(N)$ (resp. $\mathbb{L}_{n_0}(N)$) to N^S and defined by

$$\gamma\rightarrow(\gamma(t_1), ..., \gamma(t_{|S|})).$$

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Now given a *smooth* function u on N^S we shall denote $\tilde{u} = u \circ \mathscr{E}_S$. We call the family of $\tilde{u}$ the family of cylindrical functions; then the following theorem holds.

1.1. WIENER THEOREM. *There exists two borelian measures $\mu_{\mathbb{P}_{n_0}(N)}$, $\mu_{\mathbb{L}_{n_0}(N)}$ uniquely defined by the fact that the following relation holds true for every cylindrical function*:

$$\int_{\mathbb{P}_{n_0}(N)} \tilde{u}(\gamma)\, \mu_{\mathbb{P}_{n_0}(N)}(d\gamma) = \int_{N^S} u(n_1, ..., n_{|S|}) \prod_{i=1}^{|S|} p_{t_i - t_{i-1}}(n_{i-1}, n_i)\, dn_i$$

and

$$\int_{\mathbb{L}_{n_0}(N)} \tilde{u}(\gamma)\, \mu_{\mathbb{L}_{n_0}(N)}(d\gamma)$$
$$= \int_{N^S} u(n_1, ..., n_{|S|})\, p_{1-t_{|S|}}(n_{|S|}, n_0) \prod_{i=1}^{|S|} p_{t_i - t_{i-1}}(n_{i-1}, n_i)\, dn_i$$

We can apply this theorem to $\mathscr{G}$ with the usual laplacian; then we get the classical Wiener measure on $\mathscr{G}$ (which is isomorphic to the Wiener measure on $\mathbb{R}^n$). The invariant euclidean metric on $\mathscr{G}$ induces on G a biinvariant riemannian metric and a biinvariant Laplacian. The Wiener measure $\mu_{\mathbb{L}_e(G)}$ therefore defines a measure on the Loop group $\mathbb{L}_e(G)$. This measure has been considered by S. Albeverio and H. Kroehn [3], I. Frenkel [9], and S. Freed [8]. Now given M, a left-G-homogeneous space, and a base point m_0, the same construction defines a measure $\mu_{\mathbb{L}_{m_0}(M)}$ on $\mathbb{L}_{m_0}(M)$.

1.2.1. PROPOSITION. *Denote by K the stabilizer of m_0, by ${}^K\mathbb{P}(G)$ the paths on G such that*

$$\gamma(1) \in K.$$

Then ${}^KP(G)$ operates on $\mathbb{L}_{m_0}(M)$ by left multiplication.

We denote by $P^1(G)$ the paths on G of *finite energy*, that is, such that

$$\mathscr{E}(\gamma) = \int_0^1 \|\gamma^{-1}(\tau)\, \dot{\gamma}(\tau)\|^2\, d\tau < +\infty.$$

1.2.2. Instead of using the intrinsic notation of Maurer–Cartan form, throughout this paper we use the same notation as if G was a group of matrices, leaving its traduction in intrinsic terms to the reader.

We shall denote by ${}^K P^1(G)$ the paths of ${}^K P(G)$ with finite energy. We shall introduce the Fréchet space

$$\mathscr{W}(\mathbb{L}_{m_0}) = \bigcap_p L^p(\mathbb{L}_{m_0}(M), \mu_{\mathbb{L}_{m_0}(M)}).$$

Now we can now state the main results of this work.

1.3. Theorem A. *The Wiener measure on $\mathbb{L}_{m_0}(M)$ is quasi invariant under the action of ${}^K P^1(G)$. More, precisely, given $\gamma \in {}^K P^1(G)$ there exists*

$$k_\gamma \in \mathscr{W}(\mathbb{L}_{m_0})$$

such that for every

$$v \in L^q(\mathbb{L}_{m_0}(M); \mu_{\mathbb{L}_{m_0}(M)}) \qquad \textit{with} \quad q > 1,$$

we have

$$\int_{\mathbb{L}_{m_0}(M)} v(\gamma l)\, \mu_{\mathbb{L}_m(M)}(dl) = \int_{\mathbb{L}_{m_0}} v(l)\, k_\gamma(l)\, \mu_{\mathbb{L}_m(M)}(dl). \tag{1.3.1}$$

1.3.2. *Furthermore, the module of quasi invariance is a differentiable function which means the differentiability of the map*

$${}^K P^1(G) \to \mathscr{W}(\mathbb{L}_{m_0}) \textit{ defined by } \gamma \to k_\gamma$$

Remark. Formally we can say that the Cameron–Martin theorem holds true. This fact was predicted already in [3].

Theorem A can be applied in particular to $M = G$. This is the starting point which leads to the following theorem.

1.4. Theorem B. *The Wiener measure on $\mathbb{L}_e(G)$ satisfies the following properties:*

1.4.1. *Symmetry (that is, invariance by the map $\gamma \to \gamma^{-1}$).*

1.4.2. *Quasi invariance by $\mathbb{L}_e^1(G)$ acting on the right or on the left and the module of quasi invariance belongs to all L^p.*

1.4.3. *Differentiability of the quasi invariance modules on the right (resp. on the left).*

1.4.4. *Quasi invariance by the action of $\mathbb{L}_e^1(G)$ by inner automorphism on $\mathbb{L}_e(G)$.*

1.4.5. *If G is simple the triviality of the actions* 1.4.3. *or* 1.4.4. *implies that $\gamma = e$ if $\gamma = \exp(tb)$, t is small enough, $b \in \mathbb{P}_0^1(\mathscr{G})$.*

We define

$$\mathbb{L}(M) = \bigcup_{m_0 \in M} \mathbb{L}_{m_0}(M)$$

$$\mathbb{L}(G) = \bigcup_{g \in G} gP_e(G).$$

Then

$$\mathbb{L}(M) \simeq M \times L_{m_0}(M)$$

$$\mathbb{L}(G) \simeq G \times L_e(G).$$

The corresponding Wiener measure will be defined by

$$\mu_{\mathbb{L}(M)} = \mu_{L_{m_0}(M)} \otimes dm_0$$

$$\mu_{\mathbb{L}(G)} = \mu_{\mathbb{L}_e(G)} \otimes dg.$$

We now have

1.5. THEOREM C. 1.5.1. *$\mu_{\mathbb{L}(G)}$ enjoys all the properties listed in Theorem B.*

1.5.2. *$\mu_{\mathbb{L}(G)}$ and $\mu_{\mathbb{L}(M)}$ are invariant under the natural action of the circle.*

1.5.3. *$\mu_{\mathbb{L}(M)}$ is quasi invariant under the left action of $\mathbb{L}^1(G)$. The module of quasi invariance is a differentiable function on $\mathbb{L}^1(G)$.*

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2. WIENER MEASURES ON $P_0(\mathscr{G})$ AND $P_e(G)$

Theorem 1.1. defines the Wiener measure $\mu_{P_0}(\mathscr{G})$ on $\mathbb{P}_0(\mathscr{G})$. Using an orthonormal basis $e_1 \cdots e_n$ of $\mathscr{G}$ then $\mathbb{P}_0(\mathscr{G})$ is isomorphic to $\mathbb{P}_0(R^n)$. We denote

$$X = \mathbb{P}_0(R^n)$$

$$\Theta = \mu_{\mathbb{P}_0}(R^n).$$

We call (X, Θ) *the numerical Wiener space.* According the Itô's ideas, (X, Θ) will be a *universal model* for all the Wiener measures listed in Theorems A, B, and C.

2.1. *Development from* $\mathbb{P}_0(\mathscr{G})$ *to* $\mathbb{P}_e(G)$

Given $x \in P_0(\mathscr{G})$ and a partition S of $[0, 1]$ we define

$$\exp_S(x) = \gamma$$

by the following relations, $\gamma(0) = e$,

$$\gamma(t_j) = \gamma(t_{j-1}) \exp(x(t_j) - x(t_{j-1})),$$
$$\gamma(t) = \gamma(t_{j-1}) \exp\left(\frac{t - t_{j-1}}{t_j - t_{j-1}} (x(t_j) - x(t_{j-1}))\right) \qquad \text{for} \quad t \in [t_{j-1}, t_j].$$

2.1.1. THEOREM. *When the mesh of S tends to zero then, Θ a.e., the limit*

$$\lim \exp_s(x) = I(x)$$

exists in the metric space $\mathbb{P}_e(G)$.

We call $x \to I(x)$ Itô's map. Itô's map is a measurable map. Itô's map is continuous only if G is abelian. We have

2.1.2. THEOREM. $I_*(\Theta) = \boldsymbol{\mu}_{\mathbb{P}_e(G)}$.

This theorem means that the numerical Wiener space is a natural model for $(\mathbb{P}_e(G), \mu_{\mathbb{P}_e(G)})$. Itô's map will be a basic tool of our work. Its study will be made by *stochastic differential equations*. We write the following Itô's stochastic differential system:

$$\begin{aligned} g_x(0) &= e \\ dg_x &= g_x(dx - a\, dt), \end{aligned} \tag{2.1.3}$$

where

$$a = -\frac{1}{2} \sum_{k=1}^{n} e_k^2 \qquad \text{(cf. 1.2.2. for meaning of } e_k^2\text{)}. \tag{2.1.4}$$

Then $g_x(t) = (\lim_S \exp_S(x))(t)$. From now on we freely use the theory of stochastic differential equations [12, 17, 20, 7].

2.1.5. PROPOSITION. *Itô's map is almost surely a bijection from X to* $\mathbb{P}_e(G)$.

Proof. Let x, x' such that

$$g_x(t) = g_{x'}(t).$$

Then

$$g_x^{-1}(t)\,dg_x(t) = g_{x'}^{-1}(t)\,dg_x(t);$$

therefore

$$dx(t) = dx'(t)$$
$$x(t) = x'(t).$$

2.2. *An Isomorphism of the Probability Space X*

By Itô's calculus

$$dg_x^{-1} = -g_x^{-1}\,dg_x\,g_x^{-1} + g_x^{-1}\,dg_x\,g_x^{-1}\,dg_x\,g_x^{-1}$$
$$g_x^{-1}\,dg_x = e_k\,dx_k - a\,dt.$$

Therefore

$$g_x^{-1}\,dg_x\,g_x^{-1}\,dg_x\,g_x^{-1} = \sum_k e_k^2\,g_x^{-1}\,dt$$

and denoting $\tilde{g}_x = g_x^{-1}$ we get

$$d\tilde{g}_x = -dx\tilde{g}_x + (a - 2a)\,\tilde{g}_x\,dt. \tag{2.2.1}$$

Define $y(t)$ by the stochastic integral

$$y(t) = -\int_0^t \mathrm{Ad}(g_x(\xi))\,dx(\xi).$$

Then $y(t)$ is a $\mathscr{G}$-valued martingale. Let $l \in \mathscr{G}^*$ $l = (l_1, ..., l_n)$; then

$$\begin{aligned} E(l^2(y(t+\varepsilon) - y(t)) \mid \mathscr{F}_t) &= E\left(\left[\int_t^{t+\varepsilon} \langle \mathrm{Ad}\ g_x(\xi))\,dx(\xi), l\rangle\right]^2\right) \\ &= E\left(\int_t^{t+\varepsilon} \sum_k [\langle \mathrm{Ad}\ g_x(\xi)e_k, l\rangle]^2\,d\xi\right) \\ &= E\left(\int_t^{t+\varepsilon} \|\mathrm{Ad}^*\ g_x(\xi)l\|_*^2\,d\xi\right). \end{aligned}$$

By the invariance of the euclidean metric on $\mathscr{G}$, Ad $g_\alpha(\xi)$ is an orthogonal transformation. Therefore this integral is equal to $\varepsilon\,\|l\|_*^2$. Therefore, using the Paul Lévy characterization of the brownian motion, we have proved the following lemma.

2.2.2. LEMMA. *The map ζ of $\mathbb{P}_0(\mathscr{G})$ in itself defined by $\zeta(x) = y$, where*

$$y(t) = -\int_0^t \mathrm{Ad}(g_x(\xi))\, dx(\xi)$$

is an isomorphism of the probability space (X, Θ).

2.2.3. LEMMA. *The element a defined in* (2.1.4) *is invariant under the adjoint action of G.*

Proof. Let U be a representation of G, $\mathbf{U}$ its differential, then

$$\{(\Delta_G U)(g)\}_{g=e} = -\mathbf{U}(a).$$

As Δ_G is invariant by inner automorphism, if we denote U^{g_0} the representation

$$U_g^{g_0} = U_{g_0} \circ U_g \circ U_{g_0^{-1}},$$

we have

$$(\Delta U^{g_0}) = (\Delta U)^{g_0}.$$

At $g = e$ we have

$$(\psi^{g_0})(e) = \psi(g_0 e g_0^{-1}) = \psi(e).$$

Therefore

$$\{(\Delta U^{g_0})\}_{g=e} = \{\Delta U\}_{g=e}$$

implies

$$U_{g_0} \mathbf{U}(a) U_{g_0}^{-1} = \mathbf{U}(a) \qquad \text{or} \qquad \mathbf{U}(\mathrm{Ad}(g_0)a) = \mathbf{U}(a).$$

This equality being true for every unitary representation implies that $\mathrm{Ad}(g_0)a = a$.

2.2.4. LEMMA. *If I denotes Itô's map then*

$$[I(x)]^{-1} = I(\zeta(x)).$$

Proof. We have by 2.2.1, 2.2.2, and 2.2.3,

$$d\tilde{g}_x = \tilde{g}_x[dy - a\, dt]$$
$$\tilde{g}_x(0) = e.$$

By the unicity of the Cauchy problem for SDE we deduce that $\tilde{g}_x = I(y)$.

Remark. It results from 2.2.4 that $\zeta^2 = \text{identity}$, a fact which can be checked easily by direct computation.

2.2.5. PROPOSITION. *$\mu_{\mathbb{P}_e(G)}$ and $\mu_{\mathbb{L}_e(G)}$ are symmetric measures.*

Proof. The first statement is a consequence of Lemmas 2.2.2 and 2.2.4. The second statement follows from the fact that $\mu_{\mathbb{L}_e(G)}$ is the conditional law of $\mu_{\mathbb{P}_e(G)}$ under the conditioning $\gamma(1) = e$, and that this conditioning is stable, $\gamma \to \gamma^{-1}$.

Remark. The proofs of this paragraph already appeared in [3].

2.3. *Quasi Invariance*

2.3.1. THEOREM. *The measure $\mu_{\mathbb{P}_e(G)}$ is left invariant under the action of $\mathbb{P}_e^1(G)$.*

We shall prove first the infinitesimal version of this statement.

2.3.2. THEOREM. *Let $b \in \mathbb{P}_0^1(\mathscr{G})$; then for every smooth cylindrical function $\tilde{u}$ we have*

$$\left\{\frac{d}{d\tau}\int_{\mathbb{P}_e(G)} \tilde{u}(\exp(\tau b)\gamma)\, \mu_{\mathbb{P}_e(G)}(d\gamma)\right\}_{\tau=0} = \int \tilde{u}(\gamma)\, K_b(\gamma)\, \mu_{\mathbb{P}_e(G)}(d\gamma),$$

where $\mathbb{P}_0^1(\mathscr{G}) = \{\gamma: [0, 1] \to \mathscr{G},\ \gamma(0) = 0,$ with $\|\gamma\|^2_{\mathbb{P}_0^1(\mathscr{G})} := \int_0^1 \|\dot{\gamma}(t)\|^2\, dt < \infty\}$ where K_b is a borelian function such that

$$E(\exp(\lambda K_b)) < +\infty \qquad \textit{for every } \lambda.$$

2.3.3. LEMMA. *Denote $\dot{b}$ the derivative of b. Define $g_{\tau,x}$ by the following stochastic differential equation:*

$$dg_{\tau,x} = \tau\dot{b}g_{\tau,x}\, dt + g_{\tau,x}(dx - a\, dt)$$
$$g_{\tau,x}(0) = e$$

and define v_τ by the ordinary differential equation

$$dv_\tau = \tau\dot{b}v_\tau\, dt$$
$$v_\tau(0) = e.$$

Then

$$g_{\tau,x} = v_\tau\, g_x.$$

Proof. By unicity of the solution of the Cauchy problem for a linear stochastic differential equation the lemma will result from the fact that the two terms satisfy the same linear stochastic differential equation. Denote $p = v_\tau g_x$. Then

$$\begin{aligned} dp &= dv_\tau \, g_x + v_\tau \, dg_x \\ &= \tau \dot{b} v g_x \, dt + v_\tau \, g_x (dx - a\, dt) \\ &= \tau \dot{b} p \, dt + p(dx - a\, dt), \end{aligned}$$

which is the same equation as the equation defining $g_{\tau,x}$.

2.3.4. LEMMA. $\{(\partial/\partial\tau) v_\tau\}_{\tau=0} = b.$

Proof. We differentiate in τ the equation defining v_τ and we get

$$d\left(\frac{\partial}{\partial\tau} v_\tau\right) = \left(\dot{b} v_\tau + \tau b \frac{\partial v_\tau}{\partial\tau}\right) dt.$$

If we take $\tau = 0$ we get

$$d\left\{\frac{\partial}{\partial\tau} v_\tau\right\}_{\tau=0} = \dot{b}.$$

2.3.5. LEMMA. *Consider the function*

$$Q(\tau, x) = \exp\left(\tau \int_0^1 (\mathrm{Ad}(g_x(t))\, dx(t) \,|\, \dot{b}(t))_{\mathscr{G}} - \frac{\tau^2}{2} \int_0^1 \|\dot{b}(t)\|_{\mathscr{G}}^2 \, dt\right).$$

Then $g_{\tau,x}$ under $\Theta(dx)$ and g_x under $Q(\tau, x)\, \Theta(dx)$ have the same laws.

Proof. We shall write the defining equation of $g_{\tau,x}$ under the form

$$dg_{\tau,x} = g_{\tau,x}[\mathrm{Ad}(g_{\tau,x}^{-1}) \dot{b} \, dt + dx - a\, dt].$$

Then we apply the Girsanov formula.

2.3.6. *Proof of Theorem* 2.3.2. Let $\tilde{u} = u \circ \mathscr{E}_S$ a cylindrical function. Then by 2.3.4,

$$\begin{aligned} &\left\{\frac{d}{d\tau} \int_{P_0(\mathscr{G})} \tilde{u}(g_{\tau,x})\, \Theta(dx)\right\}_{\tau=0} \\ &\quad = \left\{\frac{d}{d\tau} \int_{P_0(\mathscr{G})} u(\tau b(t_1)\, g_x(t_1), ..., \tau b(t_{|S|})\, g_x(t_{|S|}))\, \Theta(dx)\right\}_{\tau=0} \\ &\quad = \left\{\frac{d}{d\tau} \int_{\mathbb{P}_0(\mathscr{G})} u(\exp(\tau b)\, g_{\tau,x})\, \Theta(dx)\right\}_{\tau=0}. \end{aligned}$$

Now applying 2.3.5, we get

$$= \int_{P_0(\mathscr{G})} \left\{ \frac{\partial}{\partial t} Q(\tau, x) \right\}_{\tau = 0} \tilde{u}(g_x)\, \Theta(dx).$$

Therefore, we get Theorem 2.3.2 with

$$K_b(x) = \int_0^1 (\mathrm{Ad}(g_x(t))\, dx(t) \mid \dot{b}(t))_{\mathscr{G}}. \tag{2.3.7}$$

Now the exponential martingale equality gives

$$E(\exp(\lambda K_b)) = \exp\left(\frac{\lambda^2}{2} \|b\|^2_{\mathbb{P}^1_0(\mathscr{G})} \right). \tag{2.3.8}$$

Remark. The above paragraph is vital for all this work. We provide an alternative proof using the stochastic calculus of variation in Section 4.

2.3.9. *Proof of Theorem* 2.3.1. We use a method of A. B. Cruzeiro [6] to reduce 2.3.1 to 2.3.2. If S_0 is a *fixed subdivision* then by the invariance of the Haar measure on G and by the strict positivity of $p_t(g)$, $k^{S_0}_{b,\tau}(\gamma)$ exists such that

$$\int_{\mathbb{P}_e(G)} \tilde{u}(\exp(\tau b)\gamma)\, \mu_{\mathbb{P}_e(G)}(d\gamma) = \int_{\mathbb{P}_e(G)} \tilde{u}(\gamma)\, k^{S_0}_{b,\tau}(\gamma)\, \mu_{\mathbb{P}_e(G)}(d\gamma)$$

for all $\tilde{u} = u \circ \mathscr{E}_{S_0}$.

Then by differentiation relative to τ and by 2.3.2 and denoting $K^{S_0}_b = E^{S_0}(K_S)$,

$$\frac{d}{d\tau} (k^{S_0}_{b,\tau})(\gamma) = k^{S_0}_{b,\tau}(\gamma)\, K^{S_0}(\gamma \exp(-\tau b)).$$

Therefore

$$k^{S_0}_{b,\tau}(\gamma) = \left(\exp\left(\tau \int_0^1 K^{S_0}_b(\exp(-\tau\lambda b)\gamma)\, d\lambda \right) \right). \tag{2.3.9.1}$$

If we take an increasing $S_0 \subset S_1 \subset S_q \subset \cdots$ family of subdivisions such that $\bigcup S_q$ is dense, then the $k^{S_q}_{b,\tau}$ will form a martingale. To prove the convergence we need to get domination of the L^p norms which will prove the convergence in L^p. It will be given by the following lemma.

2.3.9.2. Lemma. $[E(|k^S_{b,\tau}|^p)]^{p-1} \leqslant 2ch(((pq\tau)^2/2)\, \|b\|^2_{P^1})$, $1/p + 1/q = 1$.

Proof. By convexity,

$$|k^S_{b,\tau}|^p \leqslant \int_0^1 \exp(p\tau K^S_b(\exp(-\tau\lambda b)\gamma)\, d\lambda.$$

Denote

$$I_S(\tau) = (E(|k^S_{b,\tau}|^p))^{1/p}.$$

Integrating the previous inequality and applying the formula of change of variable, we get

$$(I_S(\tau))^p \leqslant E\left(\int_0^1 \exp(p\tau K^S_b)\, k^S_{b,-\tau\lambda}\, d\lambda\right).$$

By the Hölder inequality,

$$E(\exp(p\tau K^S_b) k^S_{b,-\tau\lambda}) \leqslant (I^*_S(\tau))[E(\exp(pq\tau K^S_b)]^{1/q},$$

where $1/p + 1/q = 1$ and where $I^*_S(\tau) = \sup_{|\tau'| \leqslant \tau} I_S(\tau')$. Therefore,

$$[I^*_S(\tau)]^p \leqslant I^*_S(\tau) \sup_{|\tau'| < \tau} [E(\exp(pq\tau' K^S_b)]$$

$$[I^*_S(\tau)]^{p-1} \leqslant \sup_{|\tau'| < \tau} E(\exp(pq\tau' K^S_b) \leqslant 2E(ch(pq\tau K^S_b).$$

As $K^S_b = E^S(K_b)$ we get, integrating term by term the entire series of ch, using (2.3.8), is less than or equal to

$$2ch\left(\frac{(pq\tau)^2}{2}\, \|b\|^2_{\mathbb{P}^1_0}\right).$$

2.4. *Some Algebraic Identities*

We finally have to go from the $\exp(P^1_0(\mathscr{G}))$ quasi invariance to the $P^1_e(G)$ quasi invariance. This is done using

2.4.1. LEMMA. *Let $\gamma \in P^1_e(G)$, then it is possible to find $b_1, ..., b_q \in P^1_0(\mathscr{G})$ such that*

$$\gamma = \gamma_1 \cdots \gamma_q \qquad \textit{with} \quad \gamma_i = \exp(b_i).$$

Proof. We shall denote by 2δ the distance of the unit of G to the cut locus. Then the $\log = \exp^{-1}$ is smooth in the ball of radius δ. We define

$t_1 = \sup\{t, d(\gamma(t), e) < \delta\}$ and $t_2 = \sup\{t > t_1;\ d(\gamma(t), \gamma(t_1)) < \delta\}$. By the continuity of γ we get a finite subdivision. Define

$$\begin{aligned}
\gamma_1(t) &= \gamma(t) && \text{if} \quad t \in [0, t_1] \\
&= \gamma(t_1) && \text{if} \quad t > t_1 \\
\gamma_2(t) &= \gamma^{-1}(t_1)\, \gamma(t) && \text{if} \quad t \in [t_1, t_2] \\
\gamma_2(t) &= \text{constant} && \text{if} \quad t \notin [t_1, t_2]
\end{aligned}$$

and so on.

We define $b_k = \log \gamma_k$ and we get the lemma.

2.4.2. Lemma. *If k_{γ_1} and k_{γ_2} exists and belong to all L^p, then $k_{\gamma_1\gamma_2}$ exists and has the same property.*

Proof. $k_{\gamma_1\gamma_2}(\gamma_1\alpha) = k_{\gamma_2}(\alpha) k_{\gamma_1}(\gamma_1\alpha)$.

2.4.3. Lemma. *If $\mu_{\mathbb{P}_e}$ is left quasi-invariant by γ then it is right quasi-invariant by γ^{-1}.*

Proof. $\int \varphi(\gamma\alpha)\, \mu_{\mathbb{P}_e}(d\alpha) = \int k_\gamma(\alpha)\, \varphi(\alpha)\, \mu_{\mathbb{P}_e}(d\alpha)$. Define $\varphi^\vee(\alpha) = \varphi(\alpha^{-1})$; then by symmetry,

$$\begin{aligned}
&\int \varphi^\vee(\alpha^{-1}\gamma^{-1})\, \mu(d\alpha) \\
&\quad = \int \varphi^\vee(\alpha\gamma^{-1})\, \mu(d\alpha) \\
&\quad = \int k_{\gamma^{-1}}(\alpha)\, \varphi^\vee(\alpha)\, \mu(d\alpha) = \int k_{\gamma^{-1}}(\alpha^{-1})\, \varphi(\alpha)\, \mu(d\alpha).
\end{aligned}$$

Therefore,

$$k^{\text{right}}_{\gamma^{-1}}(\alpha) = k^{\text{left}}_{\gamma}(\alpha^{-1}). \tag{2.4.4}$$

2.4.5. Theorem. *Define for $t \in [0, 1]$ $\mu^t_{\mathbb{P}_e}$ as the law of $g_x(t\tau)$, $\tau \in [0, 1]$, then $\{\mu^t_{\mathbb{P}_e}\}$ generates a convolution semi-group*:

$$\mu^{t_1}_{\mathbb{P}_e} * \mu^{t_2}_{\mathbb{P}_e} = \mu^{t_1+t_1}_{\mathbb{P}_e} \qquad \textit{if} \quad t_1 + t_2 \leqslant 1.$$

Proof. We choose on $\mathscr{G}$ two independent Brownian motions x_1 and x_2 and we consider the solution of the two following stochastic differential equations:

$$dg_{x_1}(\tau)=g_{x_1}(\tau)(\sqrt{t_1}\,dx_1(\tau)-t_1 a\,d\tau)$$
$$dg_{x_2}(\tau)=g_{x_2}(\tau)(\sqrt{t_2}\,dx_2(\tau)-t_2 a\,d\tau)$$
$$g_{x_1}(0)=g_{x_2}(0)=e.$$

Define

$$g_3(\tau)=g_{x_1}(\tau)\,g_{x_2}(\tau);$$

the stochastic differential of g_3 is

$$\begin{aligned} dg_3&=g_{x_1}(\sqrt{t_1}\,dx_1-t_1 a\,d\tau)\,g_{x_2}+g_{x_1}\,g_{x_2}(\sqrt{t_1}\,dx_2-t_1 a\,d\tau)\\ &=g_3\{\mathrm{Ad}(g_{x_2}^{-1})(\sqrt{t_1}\,dx_1-t_1 a\,d\tau)+\sqrt{t_2}\,dx_2-t_2 a\,d\tau)\}.\end{aligned}$$

Now

$$y(\tau_0)=\int_0^{t_0}\mathrm{Ad}(g_{x_2}^{-1}(\tau))\,dx_1(t)$$

is a new Brownian by 2.2. Using 2.3 we get

$$dg_3=g_3(\sqrt{t_1}\,dy+\sqrt{t_2}\,dx_2-(t_1+t_2)a).$$

Now $\sqrt{t}\;y+\sqrt{t_2}\,x_2=\sqrt{t_1+t_2}\,x_3$, where x_3 is a new Brownian. Therefore,

$$dg_3=g_3(\sqrt{t_1+t_2}\,dx_3-(t_1+t_2)a),$$

which proves the theorem.

2.5. *Differentiability of the module of quasi invariance*

2.5.1. Proposition. *The module map*

$$b\to K_b$$

is a differentiable map from $P_0^1(\mathscr{G})$ to the Frechet space $\mathscr{W}(\mathbb{P}_e)$.

Proof. We use the expression (2.3.7), which shows that, for fixed x, K_b is *a linear form* in b and therefore is differentiable.

2.5.2. Theorem. *For fixed b, the function K_b is differentiable on the right or on the left, following $P_0^1(\mathscr{G})$ vectors.*

Proof. Looking to (2.3.7), we use the fact that K_b is a stochastic integral in x and it is therefore differentiable in x according a result of D. Stroock [20], in the sense of the stochastic calculus of variation.

Now differentiability on the left of K_b corresponds, according 2.3.5, to differentiability on X following the vector field

$$\int_0^1 \mathrm{Ad}(g_n^{-1}(\xi))\,\dot{h}(\xi)\,d\xi$$

(see also 3.2.1 below). The differentiability on the right results from (2.4.4).

2.5.3. Proposition. *The module map $\gamma \to k_\gamma$ is a differentiable map $\mathbb{P}_e^1(G)$ to the Frechet space $\mathscr{W}(\mathbb{P}_e)$.*

First we define $\tilde{k}_b = k_{\exp(b)}$, $b \in \mathbb{P}_0^1(\mathscr{G})$

$$\tilde{k}_b(\gamma) = \exp\left(\int_0^1 K_b(\exp(-\lambda b)\gamma)\,d\lambda\right). \tag{2.5.3.1}$$

This formula results from (2.3.9.1) by taking the limit on an increasing sequence of partitions.

The exponential map of $P_0^1(\mathscr{G})$ to $P_e^1(G)$ is differentiable. Therefore, taking advantage of 2.4.1, we only have to prove that the map from $\mathbb{P}_0^1(\mathscr{G})$ to $\mathscr{W}(\mathbb{P}_e)$, defined by

$$b \to \tilde{k}_b,$$

is differentiable. We change in (2.5.3.1), $b \to b + \varepsilon h$; we differentiate in ε; and finally 2.5.1 and 2.5.2 give the result.

3. Localization from Paths to Loops

In the preceding section we have proved the content of Theorem B in the case of $P_e(G)$. We now have to replace $\mathbb{P}_e(G)$ by $\mathbb{L}_e(G)$. One possibility will be to use the theory of stochastic integral, the endpoint being conditioned (cf. K. Itô [13], Jeulin and Yor [14]). The quasi-sure stochastic analysis [19], [2] will provide another tool. We experiment with this tool in the present context. It is clear that it can be applied more generally to any "finite codimension conditioning."

3.1. *Quasi Sure Analysis*

On the Wiener space X we consider $P_0^1(\mathscr{G}) = H$, the Cameron–Martin space. The real-valued polynomial on X will be differentiated only *in the H-direction.* We define for a polynomial Q its $W^{p,r}$ norms by

$$\|Q\|_{W^{p,0}} = \|Q\|_{L^p(X,R)}$$
$$\|Q\|_{W^{p,1}} = \|Q\|_{L^p(X,R)} + \|\nabla Q\|_{L^p(X,H)},$$

where $(\nabla Q)(x)$ denotes the vector of H defined by

$$((\nabla Q)(x) \mid z)_H = \lim_{\varepsilon \to 0} \frac{Q(x+\varepsilon z) - Q(x)}{\varepsilon}.$$

We define

$$\|Q\|_{W^{p,2}} = \|Q\|_{L^p(X,R)} + \|\nabla^2 Q\|_{L^p(X,H \otimes H)},$$

where $H \otimes H$ is the Hilbert space having for a norm the Hilbert–Schmidt norm. In the same way we defined

$$\|Q\|_{W^{p,r}} = \|Q\|_{L^p(X,\mathbb{R})} + \|\nabla^r Q\|_{L^p(X;H \otimes \cdots \otimes H)}.$$

The space $W^{p,r}(X)$ will be the completion of the space of polynomials under the $\| \quad \|_{W^{p,r}}$ norm. It can be proved that $W^{p,r}(X)$ is a Banach space of functions of $L^p(X)$ and that $\nabla, \nabla^2 \cdots \nabla^r$ have a natural prolongation to $W^{p,r}(X)$ [2, 20]. We denote $\mathscr{W}^\infty(X) = \bigcap W^{p,r}(X)$.

3.1.1. We shall define [2] as *the capacity of an open set O*,

$$C_{p,r}(O) = \inf\{\|u\|_{W^{p,r}};\ 0 < u < 1,\ u(x) = 1 \text{ a.e. } x \in O\}.$$

A set F is said slim if

$$C^*_{p,r}(F) = 0 \qquad \text{for all } p,\ r.$$

Quasi sure analysis means statements which are true outside a slim set.

3.1.2. Redefinition. Let f be a measurable function on X. We say that f can be *redefined* if there exists a decreasing sequence O_n of open sets with *a slim intersection* and a measurable function f^* equal to f almost everywhere, such that the restriction of f^* to O_n^c *is continuous*. We shall say then that f^* is a *redefinition* of f. Two such redefinitions coincide outside a slim set.

Proposition. *If $f \in \mathscr{W}^\infty(X)$, then f can be redefined.*

3.1.2. *Non-degenerated map.* Consider the maps $X \to V$, where V is a compact finite dimension manifold; then $\mathscr{W}^\infty(X; V)$ can be defined. Given $f \in \mathscr{W}^\infty(X; V)$ then $(\nabla f)(x)$ can be defined and $\nabla f(x) \in \mathscr{L}(H; T_{f(x)}(V))$. Choose a riemannian metric on V; then we can consider $\nabla f(x) \circ (\nabla f)^*(x)$ and we shall denote

$$\det(f'(x)) = \sqrt{\det(\nabla f(x) \circ (\nabla f)^*(x)}.$$

We say that f is *non-degenerated* if $(\det(f'(x)))^{-1} \in \mathscr{W}^\infty$.

3.1.3. THEOREM. *Let $f \in \mathscr{W}^\infty(X; V)$, f non-degenerated; then there exist $k \in C^\infty(V)$ and, for every $v \in V$, a probability borelian measure on X, ρ_v such that, for all $v \in \mathscr{W}^\infty(X; R)$, we have*

$$\int_X \psi(f) u \, d\Theta = \int_V \psi(v)\, k(v)\, dv \int_{(f^*)^{-1}(v)} u^*(x)\, \rho_v(dx).$$

Furthermore, there exist (p, r) such that there exists a constant c such that

$$\left| \int u^*(x)\, \rho_v(dx) \right| < c\, \|u\|_{W^{p,r}}$$

$$\left| \int u^*(x)\, \rho_v(dx) - \int u^*(x)\, \rho_v(dx) \right| \leqslant c\, \|u\|_{W^{p,r}}\, d(v, v'),$$

where $d(v, v')$ denotes the riemannian distance on V.

Proof [2]. A consequence of this statement is the following Principle of localization.

Let (P) a property on X which is true Θ almost everywhere; then by specializing (P) to $\mathscr{W}^\infty(X)$ we get a property P_v which for all $v \in V$ is true ρ_v almost everywhere.

3.2. THEOREM. *Consider the map f from $X = P(\mathscr{G})$ to G defined by*

$$f(x) = g_x(1).$$

Then

$$f \in \mathscr{W}^\infty(X; G), \qquad \det(f'(x)) = 1;$$

f is non-degenerated.

Proof. See [10], [1] for a study of $\mathbb{L}(M)$ in gaussian geometry. See also [9] for a different approach. Choose a basis $e_1, ..., e_q$ of $\mathscr{G}$, and consider the left invariant vector fields associated $\tilde{e}_1, ..., \tilde{e}_q$. Then $\tilde{e}_j$ are smooth vector fields on G and g_x can be considered as the solution of the following stochastic differential equation

$$dg_x = \tilde{e}_j\, dx^j - \tilde{a}_g\, d\tau,$$

where $\tilde{a}_g = ga$. Then by [19, 20] we know that $f \in \mathscr{W}^\infty$.

3.2.1. LEMMA. $f'(x) \cdot h = (\int_0^1 \mathrm{Ad}(g_x(t)\, \dot{h}(t)\, dt)\, f(x)$.

Proof. This is proved in [16, part IV]. We repeat below this proof.

$$\begin{aligned} dg_{x+\varepsilon h} &= g_{x+\varepsilon h}(dx + \varepsilon \dot{h}\, dt - a\, dt) \\ &= \varepsilon \operatorname{Ad}(g_{x+\varepsilon h})\dot{h} g_{x+\varepsilon h}\, dt + g_{x+\varepsilon h}(dx - a\, dt). \end{aligned}$$

Denote $v = \{\partial g_{\varepsilon,x}/\partial\varepsilon\}_{\varepsilon=0}\, g_x^{-1}$. Then

$$dv = \operatorname{Ad} g_x \dot{h}.$$

3.2.2. *End of the Proof of Theorem* 3.2. Now if $z \in \mathscr{G}^*$ we have

$$\langle (f'(x))^* z, h\rangle = \left\langle z, \int_0^1 \operatorname{Ad}(g_x(\xi))\, h(\xi)\, d\xi \right\rangle,$$

which means

$$(f'(x))^* z = \operatorname{Ad}^*(g_x(\xi))z, \qquad \xi \in [0, 1].$$

Therefore,

$$(f'(x))(f'(x))^* = \left(\int_0^1 \operatorname{Ad}(g_x(\xi) \operatorname{Ad}^*(g_x(\xi))\, d\xi\right) z.$$

As $\operatorname{Ad}(g)$ is orthogonal we get

$$f'(x)(f'(x))^* = \text{Identity}.$$

3.4. *Proof of Theorem* B

Let $\gamma \in \mathbb{L}_e^1(G)$ then $\gamma \in \mathbb{P}_e^1(G)$ and we have

$$\begin{aligned} &\int u(\gamma\alpha)\, \psi(f(\gamma\alpha))\, \mu_{\mathbb{P}_e(G)}(d\alpha) \\ &\qquad = \int k_\gamma(\alpha)\, \psi(f(\alpha))\, u(\alpha)\, \mu_{\mathbb{P}_e}(d\alpha). \end{aligned} \tag{3.4.1}$$

We remark that

$$f(\gamma\alpha) = f(\alpha).$$

Suppose now that $u \in \mathscr{W}^\infty(X)$, $\psi \in C^\infty(G)$. Consider the redefinition u^*, f^*, k_γ^*. We denote $u^\gamma(\alpha) \equiv u(\gamma\alpha)$.

3.4.2. LEMMA. $(u^\gamma)^* = (u^*)^\gamma$.

Proof. The mapping $\gamma \to u^\gamma$ operates continuously on $\mathscr{W}^\infty(X)$. In fact,

$$\|\tau_\gamma f\|_{L^p}^p = \int f^p k_\gamma \, d\mu_{P_e} \leqslant \|f\|_{L^{2p}}^p \|k\|_{L^2}.$$

We can compute the derivative on the right ∇^r and we have the commutation

$$\nabla^r(u^\gamma) = (\nabla^r u)^\gamma.$$

As the map $\tau_\gamma : \alpha \to \gamma\alpha$ is a homeomorphism preserving the class of $\mu_{\mathbb{P}}$-null sets. Therefore τ_γ sends slim set into slim set. Given a sequence O_n of open set such that $\bigcap O_n$ is slim, then $\tau_\gamma(O_n)$ will have the same property and the lemma is proved.

We shall now decompose the formula of (3.4.1) by the desintegration 3.1.3, we get $\int_G \psi(g)\, p_1(g)\, dg \int (u^\gamma)^* (\alpha)\, \sigma_g(d\alpha) = \int_G p_1(g)\, \psi(g)\, dg \int k_\gamma^*(\alpha)$ $u^*(\alpha)\, \sigma_g(d\alpha)$. This being true for every ψ, we get *for almost all* g,

$$\int (u^\gamma)^* (\alpha)\, \sigma_g(d\alpha) = \int k_\gamma^*(\alpha)\, u^*(\alpha)\, \sigma_g(d\alpha). \tag{3.4.3}$$

By the disintegration lemma, the two member are *continuous function* of g. Therefore, the equality (3.4.3) is true for all g in particular for $g = e$. We then get

$$\int (u^\gamma)^* (\alpha)\, \mu_{\mathbb{L}_e(G)}(d\alpha) = \int u^*(\alpha)\, k_\gamma^*(\alpha)\, \mu_{\mathbb{L}_e(G)}(d\alpha)$$

or, by 3.4.2,

$$\int (u^*)^\gamma (\alpha)\, \mu_{\mathbb{L}_e(G)}(d\alpha) = \int u^*(\alpha)\, k_\gamma^*(\alpha)\, \mu_{\mathbb{L}_e(G)}(d\alpha).$$

If we take for u a cylindrical function v we get then $u^* = v$ and

$$\int v(\gamma\alpha)\, \mu_{\mathbb{L}_e(G)}(d\alpha) = \int k_\gamma^*(\alpha)\, u(\alpha)\, \mu_{\mathbb{L}_e(G)}(d\alpha).$$

As cylindrical functions are weakly dense in $C_b(\mathbb{L}_e(G))$, we get the quasi invariance.

3.5. *End of the Proof of Theorem* B

The inner automorphism quasi invariance results from the combination of the right and the left quasi invariance.

To get 1.4.5, we prove the remark that

$$K_b(g) = \int_0^1 (\mathrm{Ad}(g_x(\xi))\, dx(\xi) \mid \dot{b}(\xi)) = 0$$

is equivalent to

$$0 = E(|K_b(g)|^2) = E\left(\int_0^1 \|\mathrm{Ad}(g_x^{-1}(\xi))\, \dot{b}(\xi)\|^2\, d\xi\right)$$

$$= \int_0^1 \int_G \|\mathrm{Ad}(g^{-1})\, \dot{b}(\xi)\|^2\, p_\xi(g)\, d\xi\, dg$$

or

$$\mathrm{Ad}(g^{-1})\, \dot{b}(\xi) = 0 \qquad \text{for all} \quad g \in G.$$

If G is simple we get $\dot{b}(\xi) = 0$ for all ξ.

To consider 1.4.4, we write $\mathrm{Ad}(g_x(\xi))\, dx(\xi)$ as $(dg)\, g^{-1}$. Then by (2.4.4) and (2.3.7) the infinitesimal invariance by inner automorphism will mean that

$$0 = \int_0^1 (dgg^{-1} - g^{-1}\, dg \mid \dot{b}(\xi)) = \text{trace} \int_0^1 (dgg^{-1} - g^{-1}\, dg)\, \dot{b}(\xi)$$

$$= \text{trace} \int dg(g^{-1}\dot{b}(\xi) - \dot{b}(\xi)\, g^{-1}).$$

Taking the expectation of the square, we get

$$\int_G \|g^{-1}\dot{b}(\xi) - \dot{b}(\xi)\, g^{-1}\|^2_{H.S.}\, p_\xi(g)\, dg = 0$$

or

$$g^{-1}\dot{b}(\xi) - \dot{b}(\xi)\, g^{-1} = 0 \qquad \text{for all} \quad g \in G$$

or

$$\dot{b}(\xi) = 0.$$

4. Loops over a Homogeneous Space

4.1. *Lifting to an Horizontal Diffusion*

Let K a subgroup of G, $M = G/K$ and m_0 the base point on M. We shall denote $\mathbb{P}_{m_0}$ the paths on M starting at m_0. We denote by $\mathscr{M}$ the orthogonal of $\mathscr{K}$ in $\mathscr{G}$. G can be considered as a principal K-bundle over M.

Denote by π the projection $G \to M$. Then

$$\pi\colon \mathbb{P}_e(G) \to \mathbb{P}_{m_0}(M).$$

We define a section of this map. We say that $\varphi \in P_e(G)$, φ smooth, is *horizontal* if

$$\varphi^{-1}(t)\,\dot{\varphi}(t) \in \mathcal{M} \qquad \text{for all} \quad t \in [0, 1].$$

We denote

$$P^{\mathcal{M}}_{e,\text{smooth}}(G) = \{\varphi \cdot \varphi \text{ smooth, } \varphi \text{ horizontal}\}.$$

Then it is a well-known fact that

$$\pi \mid \mathbb{P}^{\mathcal{M}}_{e,\text{smooth}} \to \mathbb{P}_{m_0,\text{smooth}}(M)$$

is a bijection. This property prolongates on Brownian curves. More precisely, we choose an orthonormal basis $e_1, ..., e_d$ of $\mathcal{M}$. Then we shall define the horizontal diffusion by the stochastic differential equation

$$dg_y = g_y\left(\sum e_k\, dy^k - \tilde{a}\, dt\right)$$
$$g_y(0) = e,$$

where $y = (y^1, ..., y^d)$ is a Brownian motion on $\mathbb{R}^d$ and where $\tilde{a} = \sum e_k^2$.

4.1.1. THEOREM. *The map*

$$\pi\colon g_y \to \pi \circ g_y$$

defines an isomorphism Γ of the probability space $\mathbb{P}^{\mathcal{M}}_e(G)$ on $P_{m_0}(M)$. As $\mathbb{P}^{\mathcal{M}}_e(G)$ is isomorphic to Y, the Wiener space on $\mathbb{R}^d$, we get a isomorphism $j\colon Y \to \mathbb{P}_{m_0}(M)$.

Proof. We can use the limit theorem [4, 19, 18, 12] which describes functorially the Wiener measure as the limit of measures carried by a smooth curve where the theorem is true. An alternative proof is in [15].

4.1.2. PROPOSITION. *Let $q \in \mathbb{P}^1_e(G)$. Define*

$$g_{\tilde{y}} = \Gamma^{-1} \circ \pi(qg_y),$$

then

$$d\tilde{g} = dy + \pi_{\mathcal{M}}(\mathrm{Ad}(g_y^{-1}q^{-1})\dot{q}q^{-1}),$$

where $\pi_{\mathcal{M}}$ is the orthogonal projection of $\mathcal{G}$ on $\mathcal{M}$.

Proof. $d(qg_y) = dgq^{-1}qg_y + qg_y(\sum_k e_k\, dy^k - \tilde{a}\, dt) = qg_y[\sum_k e_k\, dy^k - \tilde{a}\, dt + \mathrm{Ad}(g_y^{-1}q^{-1})\dot{q}q^{-1}\, dt]$. We shall compute by Itô's calculus the stochastic differential of $\pi(qg_y)$. The first-order term appears in the given formula. In order to compute the stochastic contraction we have to compute a second-order derivation which is given by the following lemma:

4.1.3. LEMMA. *Let $m \in M$, $g \in G$ such that $\pi(g) = m$ consider on G the left invariant chart. Consider on M the chart $h \to \pi(g\exp(h))$ with $h \in \mathcal{M}$. Then $\tilde{\pi}$ will denote π read in these two charts; then*

$$\tilde{\pi}''(g_0)(h, h) = 0 \qquad \textit{for all} \quad h \in \mathcal{M}.$$

Proof. By definition of $\tilde{\pi}$ the relation $h_2 = \pi(h_1)$ is equivalent to

$$g\exp(h_2)k = g\exp(h_1) \qquad \text{with} \quad h_1 \in \mathcal{M},\ h_2 \in \mathcal{M},\ k \in \mathcal{K}.$$

The second derivative of $\tilde{\pi}$ is computed taking $h_1 = \varepsilon h_3$ and developing $h_2(\varepsilon)$ of order 2 defined by the relation

$$\exp(h_2(\varepsilon))\, k(\varepsilon) = \exp(\varepsilon h_3).$$

We write $k(\varepsilon) = \exp(s(\varepsilon))$ and we get

$$\exp(h_2(\varepsilon)) = \exp(\varepsilon h_3)\exp(-s(\varepsilon)))$$
$$h_2(\varepsilon) = \varepsilon h_3 - s(\varepsilon) + \tfrac{1}{2}\varepsilon[h_3, s(\varepsilon)] + o(\varepsilon^2).$$

We write

$$h_3 = m + k \qquad \text{with} \quad m \in \mathcal{M},\ k \in \mathcal{K}.$$

We get finally

$$s(\varepsilon) = \varepsilon k + o(\varepsilon)$$
$$h_2(\varepsilon) = \varepsilon m + \frac{\varepsilon^2}{2}\pi_{\mathcal{M}}[m, k].$$

Therefore,

$$\tilde{\pi}''(g_0)(h_3, h_3) = \pi_{\mathcal{M}}[m, k].$$

If $h_3 \in \mathcal{M}$ then $k = 0$ and the lemma is proved.

4.2. *Proof of the Quasi Invariance on $P_{m_0}(M)$*

Define for every $z \in \mathbb{P}_0^1(\mathscr{G})$,

$$\sum_{k=1}^{d} \int_0^1 (\mathrm{Ad}(g_y^{-1})z \mid e_k)\, dy^k = \hat{K}_z(y).$$

Using the isomorphism Theorem 4.1.1, we define

$$K_z(\alpha) = \hat{K}_z(j^{-1}(\alpha)).$$

4.2.1. THEOREM. *Given $z \in \mathbb{P}_0^1(\mathscr{G})$ then there exists K_z such that for every cylindrical function $\tilde{u}$ we have*

$$\left\{\frac{d}{d\varepsilon}\int_{\mathbb{P}_{m_0}} \tilde{u}(\exp(\varepsilon z)\alpha)\, \mu_{P_{m_0}}(d\alpha)\right\}_{\varepsilon=0} = \int_{\mathbb{P}_{m_0}} \tilde{u}(\alpha)\, K_z(\alpha)\, \mu_{\mathbb{P}_{m_0}}(d\alpha).$$

Furthermore, for all $\lambda \in \mathbb{R}$,

$$\int_{\mathbb{P}_{m_0}} \exp(\lambda K_z(\alpha))\, \mu_{\mathbb{P}_{m_0}}(d\alpha) < +\alpha.$$

Proof. It would be possible to prove this theorem by Girsanov formula. As in Section 2, we prefer to sketch an alternative approach which could also have been used for $\mathbb{P}_e(G)$. This approach will be based on Gaussian geometry on Y. It will give only the infinitesimal quasi invariance and Girsanov have to be invoked to integrate fully the infinitesimal action.

4.2.2. PROPOSITION. *The map j^{-1} is differentiable in the sense of the stochastic calculus of variation. Furthermore, given $z \in P_0^1(\mathscr{G})$, there exists a unique vector field Z on Y such that*

(i) $j'(y)Z_y = zj(y)$. *Furthermore,*

(ii) $\|Z\|_{W^{1,p}} \leqslant \|z\|_{\mathbb{P}^1(G)}$ *and Z_y is given by* $Z_y(t) = \pi_{\mathscr{M}}(\int_0^t \mathrm{Ad}(g^{-1}(\xi))\, \dot{z}(\xi)\, d\xi)$.

Proof. Define $q = \exp(\varepsilon z)$; then

$$d\tilde{y}_\varepsilon = dy + \pi_{\mathscr{M}}(\varepsilon\, \mathrm{Ad}(g_y^{-1}\exp(-\varepsilon z)\dot{z})).$$

Therefore,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\,[\dot{\tilde{y}}_\varepsilon - \dot{y}_0] = \pi_{\mathscr{M}}(\mathrm{Ad}(g_y^{-1})\dot{z}).$$

This formula proves the differentiability of j^{-1}. Its differential is given by

$$(j^{-1})'\,(g_y)(zg_y) = \pi_{\mathscr{M}}(\mathrm{Ad}(g_y^{-1})\dot{z}).$$

This *differential is a bijection*: In fact,

$$\pi_{\mathcal{M}}(\mathrm{Ad}(g_y^{-1})\dot{z}) = 0$$

implies

$$g_y \dot{z} \in K$$

and therefore $d\pi(g_y z) = d\pi(g_y)$ and $\pi(g_y z) = \pi(g_y)$.

As j is a bijection, $(j^{-1})'$ exists and is bijective; this implies the existence of $j'(y)$ and proves (i).

The existence of $j'(y)$ can also be proved directly as follows: Given $h \in H$, consider

$$dg_{y+\varepsilon h} = g_{y+\varepsilon h}(dy + \varepsilon \dot{h}\, dt - \tilde{a}\, dt).$$

Denote

$$\left\{\frac{\partial g_{y+\varepsilon h}}{\partial \varepsilon}\right\}_{\varepsilon=0} = k g_y$$

then

$$d(kg_y) = kg_y[dy - \tilde{a}\, dt] + g_y \dot{h}\, dt$$
$$\dot{k} g_y + k\, dg_y = kg_y[dy - \tilde{a}\, dt] + g_y \dot{h}\, dt$$
$$\dot{k} = \mathrm{Ad}(g_y)\dot{h},$$

or

(iii) $k(t) = \int_0^t \mathrm{Ad}(g_y(\xi))\, \dot{h}(\xi)\, d\xi$. Therefore,

$$\left\{\frac{\partial}{\partial \varepsilon} \mathrm{Ad}(g_{y+\varepsilon h_1}^{-1})\right\}_{\varepsilon=0} = \mathrm{Ad}(g_y^{-1})\, \mathrm{ad}(k_1).$$

We then deduce

$$(\nabla Z)(y)(t) = \pi_{\mathcal{M}} \int_0^t \mathrm{Ad}(g_y^{-1}(\xi))\, \mathrm{ad}\left(\int_0^\xi \mathrm{Ad}(g_y(\eta))\, \dot{h}_1(\eta)\, d\eta)\, \dot{z}(\xi)\, d\xi\right)$$

$$(\nabla Z_y)(h_1, h_2) = \int_0^1 (\dot{h}_2(t) \,|\, \mathrm{Ad}\, g_y^{-1}(t))$$
$$\times\, \mathrm{ad}\left(\int_0^t \mathrm{Ad}(g_y(\eta))\, \dot{h}_1(\eta))\, \dot{z}(t)\, dt\, d\eta\right).$$

Therefore,

$$\|\nabla Z_y\|_{H\otimes H} \leqslant \int_0^1 \int_0^1 \|\pi_{\mathcal{M}} \operatorname{Ad}(g_y^{-1}(t))\, \dot{z}(t)\|^2 \, dt\, d\eta \leqslant \|z\|^2_{P_1(\mathscr{G})},$$

the inequality which proves (ii).

4.2.3. *Estimations of the divergence.* We denote by δZ the divergence of the vector field Z which is defined by the identity

$$\int_Y u\, \delta z \, d\mu_Y = \int (\nabla u \mid Z)_H \, d\mu_Y$$

for all $u \in W^{1,q}(X)$. Then it is known that if $Z \in W^{1,p}$ then δZ exists and

(i) $\|\delta Z\|_{L^p} \leqslant p \, \|Z\|_{W^{1,p}}$.

We want to deduce from (i) that $\mu_{P_{m_0}}$ is infinitesimally quasi invariant under the action of $\mathbb{P}^1(\mathscr{G})$. To integrate this action we will need *exponential estimates* on the divergence. Integrating term by term the entire expansion associated to the exponential function we get that, if the right side of the following formula is positive that

$$E(\exp(\lambda\, \delta Z)) \leqslant 1 \Big/ \left(1 - \frac{\lambda}{e} \|Z\|_{W^{1,p}}\right).$$

By this formula we obtain quasi invariance by exp(b) with $\|b\|_{P^1(\mathscr{G})} \leqslant c$, c small enough.

4.2.4. *Expression of δZ by a stochastic integral.* As Z_y is an adapted vector field for the Itô's filtration then we have

$$(\delta Z_y) = \sum_{k=1}^{d} \int_0^1 (\operatorname{Ad}(g_y^{-1}(\xi))\, \dot{z}(\xi) \mid e_k)_{\mathscr{G}} \, dx^k(\xi).$$

Therefore,

$$\begin{aligned} E(\exp(\lambda\, \delta Z_y)) &= \exp\left(\frac{\lambda^2}{2} \int_0^1 \|\pi_{\mathcal{M}} \operatorname{Ad}(g_y^{-1}(\xi))\, \dot{z}(\xi)\|^2_{\mathscr{G}} \, d\xi\right) \\ &\leqslant \exp\left(\frac{\lambda^2}{2} \|z\|_{P^1(\mathscr{G})}\right). \end{aligned}$$

With this estimate the $\mathbb{P}^1_e(G)$-quasi invariance is proved by Cruzeiro's inequality [6].

4.3. *A non-degenerated Map*

Denote by Φ the map $Y \to M$ defined

$$\Phi(y) = \pi(g_y).$$

4.3.1. PROPOSITION. *We have* $\Phi \in \mathscr{W}^\infty(Y; G)$; *furthermore* Φ *is non-degenerated.*

Proof. Following 4.2.2 (iii) we have

$$\left\{\frac{\partial}{\partial \varepsilon} g_{y+\varepsilon h}(1)\right\}_{\varepsilon=0} = \left(\int_0^1 \mathrm{Ad}(g_y(\xi))\, \dot{h}(\xi)\, d\xi\right) \Phi(y).$$

We fixed h and we defined

$$p(y) = \int_0^1 \mathrm{Ad}(g_y(\xi))\, \dot{h}(\xi)\, d\xi.$$

4.3.1.1. LEMMA. $p \in \mathscr{W}^\infty(Y; \mathscr{G})$.

Proof. $|p(y)| \leqslant \|h\|_H$,

$$(\nabla p \mid h_1) = \iint [\mathrm{Ad}(g_x(\eta_1))\, \dot{h}_1(\eta_1), \mathrm{Ad}(g_x(\xi))\, \dot{h}(\xi)]\, \mathbb{1}_{\eta_1 < \xi}\, d\xi\, d\eta_1$$

$$\begin{aligned}(\nabla^2 p)(h_1, h_2) = \iiint & \mathbb{1}_{\eta_2 < \eta_1 < \xi} [[\mathrm{Ad}\, g_x(\eta_2)\, \dot{h}_2(\eta_2), \mathrm{Ad}(g_x(\eta_1))\, \dot{h}(\eta_1)], \\ & \mathrm{Ad}(g_x(\xi))\, \dot{h}(\xi)] \\ & + \mathbb{1}_{\eta_1 < \xi} \mathbb{1}_{\eta_2 < \xi} [[\mathrm{Ad}(g_x(\eta_1))\, \dot{h}_1(\eta_1), \\ & \mathrm{Ad}(g_x(\eta_2))\, \dot{h}_2(\eta_2)], \mathrm{Ad}\, g_x(\xi)\, \dot{h}(\xi)]\, d\xi\, d\eta_1\, d\eta_2.\end{aligned}$$

The second derivative is expressed as a sum of bracket with three terms. More generally the nth derivative will be composed of q_n brackets each bracket having $(n+1)$ terms. The derivative of each of these brackets will produce $(n+1)$ terms. Therefore,

$$q_{n+1} = (n+1) q_n$$

or

$$q_n = n!$$

As Ad is a unitary operator we get

$$|(\nabla^n p)(h_1, ..., h_n)| \leqslant n!\, c^n \int_{[0,1]^n} |h_1| \cdots |h_n|,$$

where $c = \sup_{z \neq 0}(1/\|z\|)\, \|\mathrm{ad}(z)\|$. Therefore,

$$\|\nabla^n p(y)\|_{H \otimes \cdots \otimes H} \leqslant n!\, c^n,$$

which proves the lemma.

We have now to prove that Φ is non-degenerated. We have

$$\begin{aligned}\Phi'(y) \cdot h &= \pi'\left(\int_0^1 \mathrm{Ad}(g_x(\xi)\, \dot{h}(\xi))\, g_y(1)\right)\\ &= \pi_{\mathscr{M}}\left(\int_0^1 \mathrm{Ad}(g_y^{-1}(1)\, g_y(\xi))\, \dot{h}(\xi)\, d\xi\right).\end{aligned}$$

Denote

$$K(g) = \pi_{\mathscr{M}}\, \mathrm{Ad}(g^{-1})\, \pi_{\mathscr{M}}\, \mathrm{Ad}(g^{-1})\pi_{\mathscr{M}}.$$

Then

$$\Phi'(y)(\Phi'(y))^* = \int_0^1 K(g_y(\xi)\, g_y^{-1}(1))\, d\xi.$$

We remark that $K(e) = I_{\mathscr{M}}$. Therefore, $\rho > 0$ exists such that $d(g, e) < \rho$ implies $K(g) > \frac{1}{2} I_{\mathscr{M}}$. Denote

$$S = \inf_{\xi} \{d(g_y(\xi)\, g_y^{-1}(1)) < \rho\}$$

$$T = 1 - S.$$

Then

$$\Phi'(y)(\Phi'(y))^* \geqslant \frac{T}{2} I_{\mathscr{M}}.$$

The non-degeneracy will therefore result from the estimate

$$E(T^{-p}) < +\infty \qquad \text{for all } p.$$

We shall interpret T as a first entrance time on this reverse process of g_y from the time 1. Define

$$\gamma(\tau) = g_y(1 - \tau).$$

Then by Doob's theory a Brownian motion z exists such that

$$d\gamma = \gamma\left(\sum e_k\, dz^k - \tilde{a}\, d\tau\right) + \nabla \log p_{1-\tau}\, d\tau.$$

We can restrict ourself to the case $t < \frac{1}{2}$. Then the drift $\nabla \log p_{1-\tau}$ is bounded in length by an absolute constant. Therefore, the stochastic differential equation of comparison for the distance [11, 17] will be of the form

$$\alpha \frac{d^2}{d\lambda^2} + \beta \frac{d}{d\lambda}, \qquad \text{when} \quad 0 < \alpha < 1,\ 0 < \beta < c.$$

We deduce from [11] that a constant $c > 0$ exists such that

$$\mathrm{Prob}(T < \varepsilon) \leqslant \exp\left(-\frac{c}{\varepsilon}\right), \qquad \varepsilon \in [0, 1].$$

4.4. *Localization from the Paths to the Loops*

We remark that

$$\mathbb{L}_{m_0}(M) = j(\Phi^{-1}(m_0)).$$

As Φ is non-degenerated, this implies that $\Phi^{-1}(m_0)$ is a submanifold of Y in the sense of gaussian geometry [2]. In this context we can write the following proposition of localization.

4.4.1. PROPOSITION. *Denote by Z a vector field on Y and by Φ a non-degenerated map. Suppose*

$$Z \in \mathscr{W}^{\infty}(Y, H)$$
$$(\nabla \Phi \mid Z) = 0.$$

Fixed $m \in M$. Denote by u_m the restriction of u to $(\Phi^*)^{-1}(m)$. Then we have

$$\int_{(\varphi^*)^{-1}(m)} u_m(y)(\delta Z)^*(y)\, \rho_m(dy) = \int_{(\varphi^*)^{-1}(m)} (\nabla u_m \mid Z^*)\, \rho_m(dy),$$

where Z^*, $(\delta Z)^*$ denotes redefinitions of Z and δZ and where u denotes any cylindrical function.

Proof. We have

$$\int_M \psi(m)\, p_1(m)\, dm \int_{(\Phi^*)^{-1}(m)} u_m(y)(\delta Z)^*(y)\, \rho_n(dy).$$

Denote

$$\tilde{\psi} = \psi \circ \Phi$$

$$= \int_Y \tilde{\psi}\, \delta Z u\, d\mu_Y$$

$$= \int_Y (Z \mid \nabla(u\tilde{\psi}))\, d\mu_Y = \int_Y \tilde{\psi}(Z \mid \nabla u)\, d\mu_Y.$$

This last equality comes from the fact that $(Z \mid \nabla\tilde{\psi}) = 0$. We therefore get

$$\int p_1(m)\, \psi(m) \int_{(\Phi^*)^{-1}(m)} (Z^* \mid \nabla u) \rho_m.$$

Now we have

$$(\nabla u \mid Z^*) = (\nabla(u_m) \mid Z^*),$$

which proves the proposition.

4.4.2. Proposition. *$\mu_{\mathbb{L}_{m_0}}$ is infinitesimally quasi invariant under the action of ${}^K\mathbb{P}^1(\mathscr{G})$. The infinitesimal module of quasi invariance belongs to all the exponential class and is differentiable.*

Proof. The quasi invariance results from 4.4.1. Then

$$\int \exp(\lambda(\delta Z)^*) \rho_{m_0} \leqslant e\, \|\exp(\lambda\, \delta Z)\|_{W^{d,2}}.$$

As δZ is smooth the right-hand side is finite if $E(\exp(3\lambda\, |\delta Z|) < \infty$ majoration which had been proved in 4.2.3.

4.5. *Proof of Theorem* A

We integrate the infinitesimal action using Cruzeiro's inequality as before.

5. Measures on Free Loops

The measure on the free loop is defined by

$$\int_{\mathbb{L}} v(\gamma)\, \mu_{\mathbb{L}}(d\gamma) = \int_G \left[\int_{\mathbb{L}_{m_0}} v(g\gamma_1)\, \mu_{\mathbb{L}_{m_0}}(d\gamma_1) \right] dg.$$

Its main new property is the following theorem.

5.1. THEOREM (S. Freed, private communication). *The measure $\mu_{\mathbb{L}(M)}$ and $\mu_{\mathbb{L}(G)}$ are invariant under the natural action of the circle.*

Proof. We consider the identification of $[0, 1[$ to the circle by the map $t \to 2\pi t$. We shall consider the partition S_q associated to the q-roots of the unit

$$S_q = \left(0, \frac{1}{q}, ..., \frac{q-1}{q}\right)$$

and a cylindrical function $\tilde{u}$ associated to this partition. Then

$$\int_{\mathbb{L}(M)} \tilde{u}\, du_{\mathbb{L}(M)} = \int_{M^q} u(m_0, m_1, ..., m_{q-1}) \prod_{i=0}^{q-1} p_{1/q}(m_i, m_{i+1})\, dm$$

with the convention $m_q = m_0$. The product being stable by circular permutation, we obtain that the integral of $\tilde{u}$ is invariant by the action of $(q+1)$-root of the unit. When $q \to \infty$ the functions $\tilde{u}$ are dense in the weak topology in $C_b(Y)$; we get them taking the theorem to the limit.

The case of G will be treated by taking $K = e$.

5.2. THEOREM. *The left action of $\mathbb{P}^1(G)$ leaves $\mu_{\mathbb{P}(M)}$ quasi invariant.*

Proof. As $\mathbb{P}^1(G)$ is *connected*, we have only to prove the statement for

$$\gamma = \exp(b), \qquad b \in \mathbb{P}^1(\mathscr{G}).$$

This means that we have to prove

5.2.1. PROPOSITION. *The infinitesimal action of $\mathbb{P}^1(\mathscr{G})$ leaves $\mu_{\mathbb{P}(M)}$ quasi invariant and the module of equivariance belongs to all the exponential class.*

Proof. Given $l \in \mathbb{P}(\mathscr{G})$, we write $a = l(0)$, $l_1 = l - l(0)$. Then $l_1 \in \mathbb{P}_0(\mathscr{G})$ and $l = l_1 + a$:

$$\begin{aligned} I &= \left\{\frac{d}{d\varepsilon}\int \tilde{u}(\exp(\varepsilon l_1 + \varepsilon a)\gamma)\, \mu_{\mathbb{P}(G)}(d\gamma)\right\}_{\varepsilon=0} \\ &= \left\{\frac{d}{d\varepsilon}\int \tilde{u}(\exp(\varepsilon l_1)\gamma)\, \mu_{\mathbb{P}(G)}(d\gamma)\right\}_{\varepsilon=0} \\ &\quad + \left\{\frac{d}{d\varepsilon}\int \tilde{u}(\exp(\varepsilon a)\gamma)\, \mu_{\mathbb{P}(G)}(d\gamma)\right\}_{\varepsilon=0}. \end{aligned}$$

By invariance of the Haar measure of G, the last integral is zero. Therefore as

$$\exp(\varepsilon l_1)\, g\gamma = g \exp(\varepsilon\, \mathrm{Ad}(g) l_1)\gamma,$$

we have

$$I=\int_G\left[\int_{\mathbb{P}_{m_0}}\tilde{u}(g\gamma)\,K_{\mathrm{Ad}(g)l_1}(\gamma)\,\mu_{\mathbb{P}_{m_0}}(d\gamma)\right]dg,$$

which proves the infinitesimal quasi invariance of $\mu_{\mathbb{P}}$, with the following module:

$$\tilde{K}_l(g\gamma)=K_{\mathrm{Ad}(g)l}(\gamma),\qquad g\in G,\ \gamma\in\mathbb{P}_{m_0}(M).$$

This module is all in the exponential class. Therefore Cruzeiro's inequality makes it possible to integrate the infinitesimal action and to obtain quasi invariance under the $\mathbb{P}^1(G)$ action. The quasi sure analysis localizes the theorem on the loops.

References

1. H. Airault and Van Bisen, *Comptes Rendus* (1990).
2. H. Airault and P. Malliavin, Intégration géométrique sur l'espace de Wiener, *Bull. Sci. Math.* **112** (1988), 3–52.
3. S. Albeverio and H. Kroehn, The Energy representation of a Sobolev Lie Group, *Compositio Math.* **36** (1978), 37–52.
4. J.-M. Bismut, "Mécanique Aléatoire," pp. 36–96, Lecture Notes in Math., Vol. 866, Springer-Verlag, New York/Berlin, 1981.
5. P. H. Cameron and W. T. Martin, Transformation of Wiener integrals under translations, *Ann. of Math.* **45** (1944), 389–396.
6. A. B. Cruzeiro, Equations différentielles sur l'espace de Wiener et formules de Cameron–Martin non linéaire, *J. Funct. Anal.* **54** (1983), 206–227.
7. D. Elworthy, Stochastic differential equation on manifolds, *in* "London Math. Soc. Lecture Note Ser., Vol. 70," Cambridge Univ. Press, Cambridge/New York, 1982.
8. S. Freed, The geometry of loops groups, *J. Differential Geom.* **28** (1988), 223–276.
9. I. B. Frenkel, Orbital theory for affine Lie algebras, *Invent. Math.* **77** (1984), 301–352.
10. E. Getzler, Dirichlet forms on Loop spaces, *Bull. Sci. Math.* **113** (1989), 151–174.
11. N. Ikeda and S. Watanabe, A comparison theorem for solutions of stochastic differential equations and its applications, *Osaka J. Math.* **14** (1977), 619–633.
12. N. Ikeda and S. Watanabe, "Stochastic Differential Equations and Diffusion Processes," 2nd ed., North-Holland, Amsterdam, 1989.
13. K. Itô, Extension of stochastic integrals, *in* "Proceedings, International Symposium or Stochastic differential equation, Kyoto, 1976" (K. Itô, Ed.), Kinokuniya, Tokyo, 1976.
14. T. Jeulin and M. Yor (Eds.), "Grossissement de filtrations: exemples et applications," Lecture Notes in Math., Vol. 1118, Springer-Verlag, New York/Berlin, 1985.
15. P. Malliavin, Parametrix trajectorielle par un opérateur hypoelliptique et repère mobile stochastique, *C.R. Acad. Sci. Paris Ser. I Math.* **281** (1975), 241–244.
16. P. Malliavin, Stochastic calculus of variation and hypoelliptic operator, *in* "Proceedings International Symposium on Stochastic Differential Equations, Kyoto, 1976" (K. Itô, Ed.), Kinokuniya, Tokyo, 1976.
17. P. Malliavin, "Géométrie différentielle stochastique," Presses Univ. Montréal, 1977.

18. P. Malliavin, Hypoellipticity in infinite dimension, *in* "Proceedings 1989 North-Western Conference" (M. Pinsky, Ed.), to appear.
19. J. Ren, Analyse quasi-sure des équations différentielles stochastiques, *Bull. Sci. Math.* (1990), 170–195.
20. S. Watanabe, "Lectures on Stochastic Differential Equations and Malliavin Calculus" (Tata Institute for Fundamental Research), Springer-Verlag, New York/Berlin, 1984.
21. D. Pickrell, Measures on infinite dimensional Grassmann manifold, *J. Funct. Anal.* **70** (1987), 323–336.
22. D. Pickrell, Separable representation for automorphism groups of infinite symmetric spaces, *J. Funct. Anal.* **90** (1990), 1–26.

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